LODAY CONSTRUCTIONS ON TWISTED PRODUCTS AND ON TORI

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Abstract. We develop a spectral sequence for the homotopy groups of Loday constructions with respect to twisted products in the case where the group involved is a constant simplicial group. We show that for commutative Hopf algebra spectra Loday constructions are stable, generalizing a result by Berest, Ramadoss and Yeung. We prove that several truncated polynomial rings are *not* multiplicatively stable by investigating their torus homology.

Introduction

When one studies commutative rings or ring spectra, important homology theories are topological Hochschild or its higher versions. These are specific examples of the Loday construction, whose definition relies on the fact that commutative ring spectra are enriched in simplicial sets: for a simplicial set X and a commutative ring spectrum R one can define the tensor $X \otimes R$ as a simplicial spectrum whose n-simplices are

$$\bigwedge_{x \in X_n} R.$$

By slight abuse of notation $X \otimes R$ also denotes the commutative ring spectrum that is the geometric realization of this simplicial spectrum. This recovers topological Hochschild homology of R, $\mathsf{THH}(R)$, when $X = S^1$, and higher topological Hochschild homology, $\mathsf{THH}^{[n]}(R)$, for higher dimensional spheres S^n. Tensoring satisfies several properties [8, VII, §2, §3], two of which are:

- If X is a homotopy pushout, $X = X_1 \cup^h_{X_0} X_2$, then the tensor product of R with X splits as a homotopy pushout in the category of commutative ring spectra which is the derived smash product:
$$(X_1 \cup^h_{X_0} X_2) \otimes R \simeq (X_1 \otimes R) \wedge^L_{(X_0 \otimes R)} (X_2 \otimes R).$$
- A product of simplicial sets $X \times Y$ gives rise to an iterated tensor product:
$$(X \times Y) \otimes R \simeq X \otimes (Y \otimes R).$$

This last expression does not, however, imply that calculating the homotopy groups of $(X \times Y) \otimes R$ is easy. In particular, if one iterates the trace map from algebraic K-theory to topological Hochschild homology n times, one obtains a map

$$K^{(n)}(R) = \underbrace{K(K(\ldots(K}_{n}(R))\ldots)) \to \underbrace{(S^1 \times \ldots \times S^1)}_{n} \otimes R.$$

Since iterated K-theory is of interest in the context of chromatic red-shift, one would like to know as much about $(S^1 \times \ldots \times S^1) \otimes R$ as possible.

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In some good cases, the homotopy type of $X \otimes R$ only depends on the suspension of X in the sense that if $\Sigma X \simeq \Sigma Y$, then one has $X \otimes R \simeq Y \otimes R$. This property is called *stability*. Stability for instance holds for Thom spectra R that arise from an infinite loop map to the classifying space $BGL_1(S)$ (see Theorem 1.1 of [25]), or for $R = KU$ or $R = KO$ [14, §4].

One can also work relative to a fixed commutative ring spectrum R and consider commutative R-algebra spectra A and ask whether $X \otimes_R A$ only depends on the homotopy type of ΣX. In this paper, we will often work with coefficients: we look at pointed simplicial sets X and place a commutative A-algebra spectrum C at the basepoint of X. In other words, when X is pointed then the inclusion of the basepoint makes $X \otimes_R A$ into a commutative A-algebra and we can look at $\mathcal{L}^R_X(A; C) = (X \otimes_R A) \wedge_A C$, the Loday construction with respect to X of A over R with coefficients in C. We call the pair $(A; C)$ *stable* if the homotopy type of $\mathcal{L}^R_X(A; C)$ only depends on the homotopy type of ΣX. Note that the ring R is not part the notation when we say that $(A; C)$ is stable although the question depends on the choice of R, so the context should specify the R we are working over. We call the commutative R-algebra A *multiplicatively stable* as in [14, Definition 2.3] if $\Sigma X \simeq \Sigma Y$ implies that $\mathcal{L}^R_X(A) \simeq \mathcal{L}^R_Y(A)$ as commutative A-algebra spectra. If A is multiplicatively stable, then for any cofibrant commutative A-algebra C, the pair $(A; C)$ is stable (see [14, Remark 2.5]).

We investigate several algebraic examples, *i.e.*, commutative ring spectra that are Eilenberg Mac Lane spectra of commutative rings. For instance we show that the pairs $(H\mathbb{Q}[t]/t^m; H\mathbb{Q})$ are not stable for all $m \geqslant 2$, extending a result by Dundas and Tenti [7]. We also prove integral and mod-p versions of this result.

Work of Berest, Ramadoss and Yeung implies that the homotopy types of $\mathcal{L}^{Hk}_X(HA; Hk)$ and $\mathcal{L}^{Hk}_X(HA)$ only depend on the homotopy type of ΣX if k is a field and if A is a commutative Hopf algebra over k. We generalize this result to commutative Hopf algebra spectra.

Moore introduced twisted cartesian products as simplicial models for fiber bundles. We develop a Serre type spectral sequence for Loday constructions of twisted cartesian products where the twisting is governed by a constant simplicial group. As a concrete example we compute the Loday construction with respect to the Klein bottle for a polynomial algebra over a field with characteristic not equal to 2.

Content. In Section 1 we recall the definition of the Loday construction and fix notation. Section 2 contains the construction of a spectral sequence for the homotopy groups of Loday constructions with respect to twisted cartesian products. Our results on commutative Hopf algebra spectra can be found in Section 3. In Section 4 we prove that truncated polynomial algebras of the form $\mathbb{Q}[t]/t^m$ and $\mathbb{Z}[t]/t^m$ for $m \geqslant 2$ are not multiplicatively stable by comparing the Loday construction of tori to the Loday construction of a bouquet of spheres corresponding to the cells of the tori. We also show that for $2 \leqslant m < p$ the $\mathbb{F}_p$-algebra $\mathbb{F}_p[t]/t^m$ is not stable.

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1. The Loday construction: basic features

We recall some definitions concerning the Loday construction and we fix notation.

For our work we can use any good symmetric monoidal category of spectra whose category of commutative monoids is Quillen equivalent to the category of E_∞-ring spectra, such as symmetric spectra [12], orthogonal spectra [17] or S-modules [8]. As parts of the paper require us to work with a specific model category we chose to work with the category of S-modules.

Let X be a finite pointed simplicial set and let $R \to A \to C$ be a sequence of maps of commutative ring spectra.

Definition 1.1. The *Loday construction with respect to X of A over R with coefficients in C* is the simplicial commutative augmented C-algebra spectrum $\mathcal{L}^R_X(A;C)$ given by

$$\mathcal{L}^R_X(A;C)_n = C \wedge \bigwedge_{x \in X_n \setminus *} A$$

where the smash products are taken over R. Here, $*$ denotes the basepoint of X and we place a copy of C at the basepoint. The simplicial structure of $\mathcal{L}^R_X(A;C)$ is straightforward: Face maps d_i on X induce multiplication in A or the A-action on C if the basepoint is involved. Degeneracies s_i on X correspond to the insertion of the unit maps $\eta_A \colon R \to A$ over all n-simplices which are not hit by $s_i \colon X_{n-1} \to X_n$.

As defined above, $\mathcal{L}^R_X(A;C)$ is a simplicial commutative augmented C-algebra spectrum. In the following we will always assume that R is a cofibrant commutative S-algebra, A is a cofibrant commutative R-algebra and C is a cofibrant commutative A-algebra. This ensures that the homotopy type of $\mathcal{L}^R_X(A;C)$ is well-defined and depends only on the homotopy type of X.

Remark 1.2. When $R \to A \to C$ is a sequence of maps of commutative rings, we can of course use the above definition for $HR \to HA \to HC$. The original construction by Loday [15, Proposition 6.4.4] used

$$C \otimes \bigotimes_{x \in X_n \setminus *} A$$

instead with the tensors taken over R as the n-simplices in $\mathcal{L}^R_X(A;C)$.

This algebraic definition also makes sense if R is a commutative ring and $A \to C$ is a map of commutative simplicial R-algebras.

It continues to work if R is a commutative ring and $A \to C$ is a map of graded-commutative R-algebras, with the n-simplices defined as above, but the maps between them require a sign correction as terms are pulled past each other—see [21, Equation (1.7.2)].

An important case is $X = S^n$. In this case we write $\mathsf{THH}^{[n],R}(A;C)$ for $\mathcal{L}^R_{S^n}(A;C)$; this is the *higher order topological Hochschild homology of order n of A over R with coefficients in C.*

Let k be a commutative ring, A be a commutative k-algebra, and M be an A-module. Then we define

$$\mathsf{THH}^{[n],k}(A;M) := \mathcal{L}^{Hk}_{S^n}(HA;HM).$$

If A is flat over k, then $\pi_*\mathsf{THH}^k(A;M) \cong \mathsf{HH}^k_*(A;M)$ [8, Theorem IX.1.7] and this also holds for higher order Hochschild homology in the sense of Pirashvili [21]: $\pi_*\mathsf{THH}^{[n],k}(A;M) \cong \mathsf{HH}^{[n],k}_*(A;M)$ if A is k-flat [4, Proposition 7.2].

Given a commutative ring A and an element $a \in A$, we write A/a instead of $A/(a)$.

2. A spectral sequence for twisted cartesian products

We will start by letting $R \to A$ be a map of commutative rings and we study Loday constructions $\mathcal{L}_B^R(A^\tau)$ over a finite simplicial set B, where τ indicates a twisting by a discrete group G that acts on A via ring isomorphisms. This construction can be adapted as in Definition 1.1 and Remark 1.2 to allow coefficients in an A-algebra C if B is pointed, and to the case where $R \to A$ is a map of commutative ring spectra, or R is a commutative ring and A is a graded-commutative R-algebra or a simplicial commutative R-algebra.

If we have a twisted cartesian product (TCP) in the sense of [18, Chapter IV] $E(\tau) = F \times_\tau B$ where the fiber F is a simplicial R-algebra and the simplicial structure group G acts on F by simplicial R-algebra isomorphisms, it is possible to generalize this definition of the Loday construction to allow twisting by a simplicial structure group, as expained in Definition 2.1 below.

We show an example where such a TCP arises: if we start with a TCP of simplicial sets $E(\tau) = F \times_\tau B$ with twisting in a simplicial structure group G acting on F simplicially on the left and with a map of commutative rings $R \to A$, we can use that twisting to construct a TCP with fiber equal to the simplicial commutative R-algebra $\mathcal{L}_F^R(A)$ and with the structure group G acting on $\mathcal{L}_F^R(A)$ by R-algebra isomorphisms. In that situation, we get that

$$\mathcal{L}_{E(\tau)}^R(A) \cong \mathcal{L}_B^R(\mathcal{L}_F^R(A)^\tau),$$

which generalizes the fact that for a product, $\mathcal{L}_{F\times B}^R(A) \cong \mathcal{L}_B^R(\mathcal{L}_F^R(A))$. If the structure group G is discrete, *i.e.*, if G is a constant simplicial group, $\mathcal{L}_{E(\tau)}^R(A)$ can be written as a bisimplicial set and we get a spectral sequence for calculating its homotopy groups.

Definition 2.1. Let B be a finite simplicial set, R be a commutative ring, and A be a commutative R-algebra (or a graded-commutative R-algebra, or a simplicial commutative R-algebra). Let G be a discrete group acting on A from the left via isomorphisms of R-algebras, and let τ be a function from the positive-dimensional simplices of B to G so that

$$\begin{array}{rcll} \tau(b) & = & [\tau(d_0 b)]^{-1}\tau(d_1 b) & \text{for } q>1, b\in B_q, \\ \tau(d_i b) & = & \tau(b) & \text{for } i\geqslant 2, q>1, b\in B_q, \\ \tau(s_i b) & = & \tau(b) & \text{for } i\geqslant 1, q>0, b\in B_q, \text{ and} \\ \tau(s_0 b) & = & e_G & \text{for } q>0, b\in B_q. \end{array} \tag{2.2}$$

The *twisted Loday construction with respect to B of A over R twisted by τ* is the simplicial commutative (resp., graded-commutative, or bisimplicial commutative) R-algebra $\mathcal{L}_B^R(A^\tau)$ given by

$$\mathcal{L}_B^R(A^\tau)_n = \mathcal{L}_{B_n}^R(A) = \bigotimes_{b\in B_n} A$$

where the tensor products are taken over R, with

$$d_0\left(\bigotimes_{b\in B_n} f_b\right) = \bigotimes_{c\in B_{n-1}} g_c \text{ with } g_c = \prod_{b:d_0 b=c} \tau(b)(f_b),$$

$$d_i\left(\bigotimes_{b\in B_n} f_b\right) = \bigotimes_{c\in B_{n-1}} g_c \text{ with } g_c = \prod_{b:d_i b=c} f_b \text{ for } 1\le i\le n, \text{ and}$$

$$s_i\left(\bigotimes_{b\in B_n} f_b\right) = \bigotimes_{d\in B_{n+1}} h_d \text{ with } h_d = \prod_{b:s_i b=d} f_b \text{ for } 0\le i\le n.$$

We should think of the copy of A sitting over a simplex $b \in B_n$ as sitting over its 0th vertex, and of $\tau(b)$ as translating between the A over b's 0th vertex and the A over b's 1st vertex.

Lemma 2.3. *The definition above makes $\mathcal{L}^R_B(A^\tau)$ into a simplicial set.*

Proof. To check this we need only check the relations involving d_0, since the ones that do not involve τ work in the same way that they do in the usual Loday construction. For $j > 1$, we get $d_0 d_j = d_{j-1} d_0$ because in both terms, for any $c \in B_{n-2}$ we get the product over all $b \in B_n$ with $d_0 d_j b = d_{j-1} d_0 b = c$ of terms that are either $\tau(b)(f_b)$ or $\tau(d_j b)(f_b)$. These are the same by the condition in Equation (2.2) above. For $j = 1$, we get the product over all $b \in B_n$ with $d_0 d_1 b = d_0 d_0 b = c$ of terms that are either $\tau(d_1 b)(f_b)$ or $\tau(d_0 b)\tau(b)(f_b)$, which again agree by Equation (2.2). We get $d_0 s_0 = \mathrm{id}$ since $\tau(s_0 b) = e_G$, and $d_0 s_i = s_{i-1} d_o$ for $i > 0$ since for those i, $\tau(s_i b) = \tau(b)$. $\square$

Following Moore, May considers the following simplicial version of a fiber bundle [18, Definition 18.3]:

Definition 2.4. Let F and B be simplicial sets and let G be a simplicial group which acts on F from the left. Let $\tau\colon B_q \to G_{q-1}$ for all $q > 0$ be functions so that

$$\begin{array}{rcll} d_0\tau(b) & = & [\tau(d_0 b)]^{-1}\tau(d_1 b) & \text{for } q > 1, b \in B_q, \\ \tau(d_{i+1} b) & = & d_i \tau(b) & \text{for } i \geq 1, q > 1, b \in B_q, \\ \tau(s_{i+1} b) & = & s_i \tau(b) & \text{for } i \geq 0, q > 0, b \in B_q, \text{ and} \\ \tau(s_0 b) & = & e_q & \text{for } q > 0, b \in B_q. \end{array}$$

The *twisted Cartesian product (TCP)* $E(\tau) = F \times_\tau B$ is the simplicial set whose n-simplices are given by

$$E(\tau)_n = F_n \times B_n,$$

with simplicial structure maps

(i) $d_0(f, b) = (\tau(b) \cdot d_0 f, d_0 b)$,
(ii) $d_i(f, b) = (d_i f, d_i b)$ $\forall i > 0$, and
(iii) $s_i(f, b) = (s_i f, s_i b)$ $\forall i \geq 0$.

These structure maps satisfy the necessary relations to be a simplicial set because of the conditions that τ satisfies.

Definition 2.5. If R is a commutative ring and $E(\tau) = C \times_\tau B$ is a TCP as in Definition 2.4 where C is a commutative simplicial R-algebra and the simplicial group G acts on C by R-algebra isomorphisms (that is, for every $q \geq 0$, the group G_q acts on the commutative R-algebra C_q by R-algebra isomorphisms) then we can use the twisting τ to define the *twisted Loday construction with respect to B of C over R, twisted by τ*,

$$\mathcal{L}^R_B(C^\tau)_n = \mathcal{L}^R_{B_n}(C_n) = \bigotimes_{b \in B_n} C_n$$

with twisted structure maps given on monomials $\bigotimes_{b \in B_n} f_b$, with $f_b \in C_n$ for all $b \in B_n$, by

$$\begin{aligned} d_0\left(\bigotimes_{b \in B_n} f_b\right) &= \bigotimes_{c \in B_{n-1}} g_c \text{ with } g_c = \prod_{b : d_0 b = c} \tau(b)(d_0 f_b), \\ d_i\left(\bigotimes_{b \in B_n} f_b\right) &= \bigotimes_{c \in B_{n-1}} g_c \text{ with } g_c = \prod_{b : d_i b = c} d_i f_b \text{ for } 1 \leq i \leq n, \text{ and} \\ s_i\left(\bigotimes_{b \in B_n} f_b\right) &= \bigotimes_{d \in B_{n+1}} h_d \text{ with } h_d = \prod_{b : s_i b = d} s_i f_b \text{ for } 0 \leq i \leq n. \end{aligned} \tag{2.6}$$

Note that there are two sets of simplicial structure maps being used, those of C inside and those of B outside. This looks like the diagonal of a bisimplicial set, but since our twisting $\tau\colon B_{q+1} \to G_q$ explains only how to twist elements in C_q, this is not the case unless the structure group G is a discrete group, viewed as a constant simplicial group.

If the structure group G is discrete, there is overlap between Definition 2.1 and Definition 2.5. The simplicial commutative R-algebra case of Definition 2.1 actually gives a bisimplicial set: we use only the simplicial structure of B in the definition and if A also has simplicial structure, that remains untouched. The diagonal of that bisimplicial set agrees with the constant simplicial group case of Definition 2.5.

Given any TCP of simplicial sets $E(\tau) = F \times_\tau B$ as in Definition 2.4 and a map $R \to A$ of commutative rings, we can construct $\mathcal{L}^R_F(A) \times_\tau B$ which is a TCP of commutative simplicial algebras R-algebras as in Definition 2.5 using the same structure group G and twisting function $\tau\colon B_q \to G_{q-1}$. We use the simplicial left action of G_n on F_n which we denote by $(g, f) \mapsto gf$ to obtain a left action by simplicial R-algebra isomorphisms

$$\begin{aligned} G_n \times \mathcal{L}^R_{F_n}(A) &\to \mathcal{L}^R_{F_n}(A) \\ (g, \bigotimes_{f\in F_n} a_f) &\mapsto \bigotimes_{f\in F_n} a_{g^{-1}f}. \end{aligned} \tag{2.7}$$

Since the original action of G_n on F_n was a left action, this is a left action. In the original monomial, the fth coordinate is a_f. After $g \in G_n$ acts on it, the fth coordinate is $b_f = a_{g^{-1}f}$. After $h \in G_n$ acts on the result of the action of g, the fth coordinate is $b_{h^{-1}f} = a_{g^{-1}h^{-1}f}$, which is the same as the result of acting by hg on the monomial.

Proposition 2.8. *If $E(\tau) = F \times_\tau B$ is a TCP and $R \to A$ is a map of commutative rings, and we use the simplicial set twisting function τ to construct a simplicial R-algebra twisting function to obtain a TCP $\mathcal{L}^R_F(A) \times_\tau B$ as above, we get that*

$$\mathcal{L}^R_{E(\tau)}(A) \cong \mathcal{L}^R_B(\mathcal{L}^R_F(A)^\tau).$$

This uses the definition of the Loday construction of a simplicial algebra twisted by a simplicial group in Definition 2.5.

Proposition 2.8 generalizes the well-known fact that for a product of simplicial sets,

$$\mathcal{L}^R_{F\times B}(A) \cong \mathcal{L}^R_B(\mathcal{L}^R_F(A)).$$

Proof. Both $\mathcal{L}^R_{E(\tau)}(A)$ and $\mathcal{L}^R_B(\mathcal{L}^R_F(A)^\tau)$ have the same set of n-simplices for every $n \geqslant 0$:

$$\bigotimes_{e\in E(\tau)_n} A = \bigotimes_{(f,b)\in F_n\times B_n} A \cong \bigotimes_{b\in B_n} (\bigotimes_{f\in F_n} A).$$

We have to show that the simplicial structure maps agree with respect to this identification.

For $1 \leqslant i \leqslant n$, for any choice of elements $x_{(f,b)} \in A$,

$$d_i\left(\bigotimes_{(f,b)\in F_n\times B_n} x_{(f,b)}\right) = \bigotimes_{(g,c)\in F_{n-1}\times B_{n-1}} y_{(g,c)}$$

where

$$y_{(g,c)} = \prod_{(f,b):(d_i f, d_i b)=(g,c)} x_{(f,b)} = \prod_{b:d_i b=c}\left(\prod_{f:d_i f=g} x_{(f,b)}\right).$$

The internal product on the right-hand side is what we get from d_i on $\mathcal{L}^R_F(A)$ and the external product is what we get from d_i of $\mathcal{L}^R_B$, so this agrees with the definition in Equation (2.6).

The proof that the s_i, $0 \leqslant i \leqslant n$ agree is very similar.

The interesting case is that of d_0. For any choice of elements $x_{(f,b)} \in A$, the boundary d_0 associated to $\mathcal{L}^R_{E(\tau)}(A)$ satisfies

$$d_0\left(\bigotimes_{(f,b)\in F_n\times B_n} x_{(f,b)}\right) = \bigotimes_{(g,c)\in F_{n-1}\times B_{n-1}} y_{(g,c)}, \tag{2.9}$$

where

$$y_{(g,c)} = \prod_{(f,b):d_0(f,b)=(g,c)} x_{(f,b)} = \prod_{b:d_0b=c}\left(\prod_{f:\tau(b)\cdot d_0f=g} x_{(f,b)}\right).$$

From the $\mathcal{L}^R_B(\mathcal{L}^R_F(A))$ point of view, by Equation (2.6).

$$\begin{aligned}
d_0\left(\bigotimes_{b\in B_n}(\bigotimes_{f\in F_n} x_{(f,b)})\right) &= \bigotimes_{c\in B_{n-1}}\prod_{b:d_0b=c}\tau(b)d_0\left(\bigotimes_{f\in F_n} x_{(f,b)}\right)\\
&= \bigotimes_{c\in B_{n-1}}\prod_{b:d_0b=c}\tau(b)\left(\bigotimes_{g\in F_{n-1}}\prod_{f:d_0f=g} x_{(f,b)}\right)\\
&= \bigotimes_{c\in B_{n-1}}\prod_{b:d_0b=c}\left(\bigotimes_{g\in F_{n-1}}\prod_{f:d_0f=\tau(b)^{-1}g} x_{(f,b)}\right)\\
&= \bigotimes_{(g,c)\in F_{n-1}\times B_{n-1}}\prod_{b:d_0b=c}\left(\prod_{f:d_0f=\tau(b)^{-1}g} x_{(f,b)}\right),
\end{aligned}$$

which is exactly what we got in (2.9). □

If G is a discrete group and $E(\tau)$ is constructed using G, then for every $q > 0$ there is a function $\tau\colon B_q \to G$ satisfying the conditions listed in Equation (2.2) and G acts simplicially on F on the left.

Theorem 2.10. *If $E(\tau) = F \times_\tau B$ is a TCP where the twisting is by a constant simplicial group G and if $R \to A$ is a map of commutative rings so that $\pi_*(\mathcal{L}^R_F(A))$ is flat over R, then there is a spectral sequence*

$$E^2_{p,q} = \pi_p((\mathcal{L}^R_B(\pi_*\mathcal{L}^R_F(A)^\tau))_q) \Rightarrow \pi_{p+q}(\mathcal{L}^R_{E(\tau)}(A)). \tag{2.11}$$

Here, $\pi_*\mathcal{L}^R_F(A)$ is a graded commutative R-algebra. For any fixed p and q, we consider the degree q part of $\mathcal{L}^R_{B_p}(\pi_*\mathcal{L}^R_F(A)^\tau)$, $(\mathcal{L}^R_{B_p}(\pi_*\mathcal{L}^R_F(A)^\tau))_q$. This forms a simplicial abelian group which in degree p is $(\mathcal{L}^R_{B_p}(\pi_*\mathcal{L}^R_F(A)))_q$, with simplicial structure maps induced by those of B with the twisting by τ, and $\pi_p((\mathcal{L}^R_B(\pi_*\mathcal{L}^R_F(A)^\tau))_q)$ denotes its pth homotopy group. The flatness assumption above is for instance satisfied if R is a field.

Proof. Since the twisting is by a constant simplicial group G, we are able to form a bisimplicial R-algebra

$$(m,n) \mapsto \bigotimes_{b\in B_m}\bigotimes_{f\in F_n} A. \tag{2.12}$$

In the n-direction, the simplicial structure maps d^F_i and s^F_i will simply be the simplicial structure maps of the Loday construction $\mathcal{L}^R_F(A)$ applied simultaneously to all the copies of $\mathcal{L}^R_F(A)$ over all the $b \in B_n$. In the m direction, d_i^B and s_i^B are the simplicial structure maps of the twisted Loday construction, as in Equation (2.2) in Definition 2.1. These commute exactly because the simplicial structure maps in G are all equal to the identity. For any choice of $x_b \in \mathcal{L}_F^R(A)_n$ for all $b \in B_m$,

$$d_0^B d_i^F\left(\bigotimes_{b\in B_m} x_b\right) = d_0^B\left(\bigotimes_{b\in B_m} d_i^F(x_b)\right) = \bigotimes_{c\in B_{m-1}}\prod_{b:d_0b=c}\tau(b)d_i^F(x_b)$$

while

$$d_i^F d_0^B(\bigotimes_{b\in B_m} x_b) = d_i^F\left(\bigotimes_{c\in B_{m-1}}\prod_{b:d_0b=c}\tau(b)\cdot x_b\right) = \bigotimes_{c\in B_{m-1}}\prod_{b:d_0b=c}d_i^F(\tau(b)\cdot x_b),$$

which is the same since

$$d_i^F(\tau(b)\cdot x_b) = d_i(\tau(b))\cdot d_i^F(x_b) = \tau(b)\cdot d_i^F(x_b).$$

Note that since the twisting is by a constant simplicial group, $\mathcal{L}_{E(\tau)}^R(A) \cong \mathcal{L}_B^R(\mathcal{L}_F^R(A)^\tau)$ is exactly the diagonal of the bisimplicial R-algebra in Equation (2.12).

We use the standard result (see for instance [9, Theorem 2.4 of Section IV.2.2]) that the total complex of a bisimplicial abelian group with the alternating sums of the vertical and the horizontal face maps is chain homotopy equivalent to the usual chain complex associated to the diagonal of that bisimplicial abelian group. Since we know that the realization of the diagonal is homeomorphic to the double realization of the bisimplicial abelian group, in order to know the homotopy groups of the double realization of a bisimplicial abelian group, we can calculate the homology of its total complex with respect to the alternating sums of the vertical and the horizontal face maps. Filtering by columns gives an E^2 spectral sequence calculating the homology of the total complex associated to a bisimplicial abelian group consisting of what we get by first taking vertical homology and then taking horizontal homology. In the case of the bisimplicial abelian group we have in Equation (2.12), the vertical qth homology of the columns will be the qth homology with respect to $\sum_{i=0}^n(-1)^i d_i^F$ of the complex

$$\bigotimes_{b\in B_m}\mathcal{L}_F^R(A)$$

and this is isomorphic to $\pi_q\left(\bigotimes_{b\in B_m}\mathcal{L}_F^R(A)\right)$. Since we assumed that $\pi_*(\mathcal{L}_F^R(A))$ is flat over R, we obtain

$$\pi_q\left(\bigotimes_{b\in B_m}\mathcal{L}_F^R(A)\right) \cong (\bigotimes_{b\in B_m}\pi_*(\mathcal{L}_F^R(A)))_q.$$

Here, the subscript q denotes the degree q part of the graded abelian group $\bigotimes_{b\in B_m}\pi_*(\mathcal{L}_F^R(A))$.

Moreover, the effect of the horizontal boundary map on $\bigotimes_{b\in B_m}\pi_*(\mathcal{L}_F^R(A))$ is the boundary of the twisted Loday construction, with the action of G on the graded-commutative R-algebra $\pi_*(\mathcal{L}_F^R(A))$ induced by that of G on the commutative simplicial R-algebra $\mathcal{L}_F^R(A)$. As the boundary map preserves internal degree, we get the desired spectral sequence. □

2.1. **Norms and finite coverings of S^1.** The connected n-fold cover of S^1 given by the degree n map can be made into a TCP as follows. Let $B = S^1$ be the standard simplicial circle and $C_n = \langle\gamma\rangle$ be the cyclic group of order n with generator γ. The twisting function $\tau\colon S_q^1 \to C_n$ sends the non-degenerate simplex in S_1^1 to γ and is then determined by Equation (2.2). Let $F = C_n$, viewed as a constant simplicial set, and let C_n act on F from the left. Then $E(\tau) = F \times_\tau B$ is in fact another simplicial model of S^1 with n non-degenerate 1-simplices. Therefore,

$$\mathcal{L}_{E(\tau)}^R(A) \simeq \mathcal{L}_{S^1}^R(A) \quad \text{and} \quad \pi_*(\mathcal{L}_{E(\tau)}^R(A)) \cong \mathsf{HH}_*^R(A)$$

for every commutative R-algebra A. In this case, $\mathcal{L}^R_F A = A^{\otimes_R n}$ is the constant commutative simplicial R-algebra, with the C_n-action given by

$$\gamma(a_1 \otimes \cdots \otimes a_n) = a_n \otimes a_1 \otimes \cdots \otimes a_{n-1}.$$

As $\mathcal{L}^R_F(A)$ is a constant simplicial object, we obtain that

$$\pi_* \mathcal{L}^R_F(A) \cong \begin{cases} A^{\otimes_R n}, & * = 0, \\ 0, & * > 0. \end{cases}$$

If A is flat over R, the spectral sequence of Equation (2.11) is

$$E^2_{p,q} = \pi_p(\mathcal{L}^R_{S^1}(A^{\otimes_R n})^\tau)_q \Rightarrow \pi_{p+q}\mathcal{L}^R_{E(\tau)}(A) \cong \mathsf{HH}^R_{p+q}(A).$$

But here, the spectral sequence is concentrated in q-degree zero, and hence it collapses, yielding

$$\pi_p(\mathcal{L}^R_{S^1}(A^{\otimes_R n})^\tau) \cong \mathsf{HH}^R_p(A).$$

With Proposition 2.8 we can identify $\mathcal{L}^S_{E(\tau)}(A)$ if A is a commutative ring spectrum and we recover the known result (see for instance [2, p. 2150]) that

$$\mathsf{THH}_{C_n}(N^{C_n}_e A) \simeq \mathsf{THH}(A). \tag{2.13}$$

Here, $\mathsf{THH}_{C_n}(A) = N^{S^1}_{C_n}(A)$ is the C_n-relative THH defined in [2, Definition 8.2], where $N^{C_n}_e A$ is the Hill-Hopkins-Ravanel norm. See also [1, Definition 2.0.1]. The identification in (2.13) is an instance of the transitivity of the norm: $N^{S^1}_{C_n} N^{C_n}_e A \simeq N^{S^1}_e A$.

2.2. **The case of the Klein bottle.** For the Klein bottle we compute the homotopy groups of the Loday construction of the polynomial algebra $k[x]$ for a field k using our TCP spectral sequence and we confirm our answer using the following pushout argument. We assume that the characteristic of k is not 2, so 2 is invertible in k.

Note that the Klein bottle can be represented as a homotopy pushout $K\ell \simeq (S^1 \vee S^1) \cup^h_{S^1} D^2$. Since the Loday construction converts homotopy pushouts of simplicial sets into homotopy pushouts of commutative algebra spectra, we obtain

$$\mathcal{L}^k_{K\ell}(k[x]) \simeq \mathcal{L}^k_{S^1 \vee S^1}(k[x]) \wedge^L_{\mathcal{L}^k_{S^1}(k[x])} \mathcal{L}^k_{D^2}(k[x]).$$

Homotopy invariance of the Loday construction yields that $\pi_*\mathcal{L}^k_{D^2}(k[x]) \cong k[x]$, and as $\pi_*\mathcal{L}^k_{S^1}(k[x]) = \mathsf{HH}^k_*(k[x])$ is well known to be isomorphic to $k[x] \otimes \Lambda(\varepsilon x)$ as a graded commutative k-algebra, we get that

$$\pi_*\mathcal{L}^k_{S^1 \vee S^1}(k[x]) \cong \pi_*\mathcal{L}^k_{S^1}(k[x]) \otimes_{k[x]} \pi_*\mathcal{L}^k_{S^1}(k[x]) \cong k[x] \otimes \Lambda(\varepsilon x_a, \varepsilon x_b)$$

where the indices a and b allow us to distinguish between the generators emerging from each of the circles $S^1_a \vee S^1_b$. Let S^1_c represent the circle along which we glue the disk, and call the corresponding generator in dimension one for the Loday construction over it εx_c. Let S^1_a denote the circle that S^1_c will go twice around in the same direction and S^1_b denote the circle that it will go around in opposite directions. So we have a projection $K\ell \to S^1_b$.

We can calculate $\pi_*\mathcal{L}^k_{K\ell}(k[x])$ with a Tor spectral sequence whose E^2-page is

$$\begin{aligned} E^2_{*,*} &= \mathrm{Tor}^{\pi_*\mathcal{L}^k_{S^1}(k[x])}_{*,*}\Big(\pi_*\mathcal{L}^k_{S^1 \vee S^1}(k[x]), \pi_*\mathcal{L}^k_{D^2}(k[x])\Big) \\ &\cong \mathrm{Tor}^{k[x]\otimes\Lambda(\varepsilon x_c)}_{*,*}(k[x] \otimes \Lambda(\varepsilon x_a, \varepsilon x_b), k[x]). \end{aligned} \tag{2.14}$$

We need to understand the $\pi_*\mathcal{L}^k_{S^1}(k[x])$-module structure on $k[x] \otimes \Lambda(\varepsilon x_a, \varepsilon x_b)$, so we need to understand the map

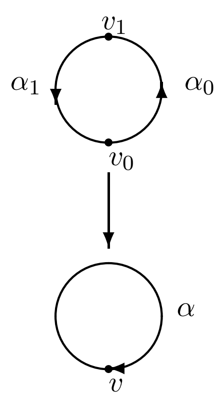
$$k[x] \otimes \Lambda(\varepsilon x_c) \to k[x] \otimes \Lambda(\varepsilon x_a, \varepsilon x_b).$$

Since $k[x]$ in both cases is the image of the Loday construction on a point, we know that x on the left maps to x on the right. If we map S^1_c to $S^1_a \vee S^1_b$ and then collapse S^1_a to a point, we end up with a map $S^1_c \to S^1_b$ that is contractible, so if we only look at the εx_b part of the image of εx_c in $\Lambda(\varepsilon x_a, \varepsilon x_b)$ (that is, if we augment εx_a to zero) we get zero.

We deduce that

$$\begin{aligned}&\mathrm{Tor}_{*,*}^{k[x]\otimes\Lambda(\varepsilon x_c)}(k[x]\otimes\Lambda(\varepsilon x_a, \varepsilon x_b), k[x])\\ &\qquad\cong \mathrm{Tor}_{*,*}^{k[x]}(k[x], k[x]) \otimes \mathrm{Tor}_{*,*}^{\Lambda(\varepsilon x_c)}(\Lambda(\varepsilon x_a), k) \otimes \mathrm{Tor}_{*,*}^{k}(\Lambda(\varepsilon x_b), k)\\ &\qquad\cong k[x] \otimes \mathrm{Tor}_{*,*}^{\Lambda(\varepsilon x_c)}(\Lambda(\varepsilon x_a), k) \otimes \Lambda(\varepsilon x_b).\end{aligned}$$

In order to calculate $\mathrm{Tor}_{*,*}^{\Lambda(\varepsilon x_c)}(\Lambda(\varepsilon x_a), k)$, we map S^1_c to $S^1_a \vee S^1_b$ and then collapse S^1_b to a point. This gives a map $S^1_c \to S^1_a$ that is homotopic to the double cover of the circle as depicted below. We consider elements of $\mathcal{L}_{S^1_c}(k[x])$, which we think of as built on the top circle, and of $\mathcal{L}_{S^1_a}(k[x])$, which we think of as built on the bottom circle, and write them as sums of tensor monomials of ring elements with subscripts indicating the simplex each ring element lies over. Under this map, we have



$$x_{\alpha_0} := 1_{s_0v_0} \otimes 1_{s_0v_1} \otimes x_{\alpha_0} \otimes 1_{\alpha_1} \mapsto 1_{s_0v} \otimes x_\alpha$$
$$x_{\alpha_1} := 1_{s_0v_0} \otimes 1_{s_0v_1} \otimes 1_{\alpha_0} \otimes x_{\alpha_1} \mapsto 1_{s_0v} \otimes x_\alpha$$

Then $d_0 - d_1$ maps these elements to the following:

$$x_{\alpha_0} \mapsto 1_{v_0} \otimes x_{v_1} - x_{v_0} \otimes 1_{v_1}$$
$$x_{\alpha_1} \mapsto x_{v_0} \otimes 1_{v_1} - 1_{v_0} \otimes x_{v_1}$$

Note that the sum of the images under $d_0 - d_1$ is zero, and so $x_{\alpha_0} + x_{\alpha_1}$ is a cycle with one copy of x in simplicial degree 1, which is what εx_c should be. Monomials that put the copy of x over s_0v_i are the image under $d_0 - d_1 + d_2$ of monomials that put one copy of x over $s_0^2 v_i$, so do not contribute to the homology, and all cycles not involving those and involving only one copy of x are multiples of $x_{\alpha_0} + x_{\alpha_1}$, and so $x_{\alpha_0} + x_{\alpha_1}$ represents εx_c. But we know that εx_a is represented by $1_{s_0v} \otimes x_\alpha$, so we get that that $\varepsilon x_c \mapsto 2\varepsilon x_a$.

We take the standard resolution of k as a $\Lambda(\varepsilon x_c)$-module:

$$\ldots \xrightarrow{\cdot\varepsilon x_c} \Sigma\Lambda(\varepsilon x_c) \xrightarrow{\cdot\varepsilon x_c} \Lambda(\varepsilon x_c)$$

Since we saw above that $\varepsilon x_c \mapsto 2\varepsilon x_a$, tensoring $(-) \otimes_{\Lambda(\varepsilon x_c)} \Lambda(\varepsilon x_a)$ yields

$$\ldots \xrightarrow{\cdot 2\varepsilon x_a} \Sigma\Lambda(\varepsilon x_a) \xrightarrow{\cdot 2\varepsilon x_a} \Lambda(\varepsilon x_a)$$

Since we assume that 2 is invertible in k, we get that $\mathrm{Tor}_*^{\Lambda(\varepsilon x_c)}(\Lambda(\varepsilon x_a), k) \cong k$, and so when 2 is invertible in k, the spectral sequence in Equation (2.14) has the form

$$E^2_{*,*} \cong k[x] \otimes k \otimes \Lambda(\varepsilon x_b) \cong k[x] \otimes \Lambda(\varepsilon x_b),$$

and therefore also collapses for degree reasons, yielding

$$\pi_*\mathcal{L}^k_{K\ell}(k[x]) \cong k[x] \otimes \Lambda(\varepsilon x).$$

Remark 2.15. In fact we have shown that $\pi_*\mathcal{L}^k_{K\ell}(k[x]) \cong \pi_*\mathcal{L}^k_{S^1}(k[x])$ and that the projection $K\ell \to S^1_b$ induces this isomorphism.

Now we want to get the same result using our TCP spectral sequence for $A = k[x]$. We will use the following simplicial model for the Klein bottle:

$$K\ell = (I \times S^1)/(0,t) \sim (1, \mathrm{flip}(t))$$

where flip is the reflection of the circle about the y-axis. If we use the same model of the circle with two vertices and two edges that we used in the double cover picture above but we reverse the orientation on α_0 so that both edges go top to bottom, this is a simplicial map preserving v_0 and v_1 and exchanging the α_i.

The flip map induces a map on $\pi_*(\mathcal{L}^k_{S^1}(k[x]) \cong k[x] \otimes \Lambda(\varepsilon x)$ sending $x \mapsto x$ and $\varepsilon x \mapsto -\varepsilon x$. The fact that $x \mapsto x$ comes from the fact that it is the image of the Loday construction over a point. Using the same notation and argument as before, with the different orientation on α_0, εx can be represented by $x_{\alpha_0} - x_{\alpha_1}$, so exchanging the α_i sends εx to $-\varepsilon x$.

The nontrivial twist $\tau\colon S^1 \to C_2 = \langle \gamma \rangle$ maps the non-degenerate 1-cell $\alpha \in S^1_1$ to γ and is then determined by Equation (2.2), yielding

$$d_0(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = a_0 \cdot \gamma a_1 \otimes a_2 \otimes \ldots \otimes a_n. \tag{2.16}$$

The TCP spectral sequence (2.11) in this case takes the form

$$E^2_{p,q} = \pi_p \left(\left(\mathcal{L}^k_{S^1} \left(\pi_*(\mathcal{L}^k_{S^1}(k[x]))^\tau \right) \right)_q \right) \implies \pi_{p+q} \mathcal{L}^k_{K\ell}(k[x])$$

and since $\pi_* \mathcal{L}^k_{S^1}(k[x]) \cong k[x] \otimes \Lambda(\varepsilon x)$,

$$E^2_{p,q} = \pi_p \left(\left(\mathcal{L}^k_{S^1} \left(k[x] \otimes \Lambda(\varepsilon x)^\tau \right) \right)_q \right),$$

which is the pth homotopy group of the simplicial k-vector space whose p-simplices are

$$\left(\mathcal{L}^k_{S^1_p}(k[x] \otimes \Lambda(\varepsilon x)^\tau) \right)_q.$$

For each p, $\mathcal{L}^k_{S^1_p}(k[x] \otimes \Lambda(\varepsilon x)) \simeq \mathcal{L}^k_{S^1_p}(k[x]) \otimes_k \mathcal{L}^k_{S^1_p}(\Lambda(\varepsilon x))$, and so $\mathcal{L}^k_{S^1}(k[x] \otimes \Lambda(\varepsilon x)^\tau) \simeq \mathcal{L}^k_{S^1}(k[x]) \otimes_k \mathcal{L}^k_{S^1}(\Lambda(\varepsilon x)^\tau)$. We can think of this tensor product of simplicial k-algebras as the diagonal of a bisimplicial abelian group, and again by [9, Theorem 2.4 of Section IV.2.2] the total complex of a bisimplicial abelian group with the alternating sums of the vertical and the horizontal face maps is chain homotopy equivalent to the usual chain complex associated to the diagonal of that bisimplicial abelian group. But in this case of a tensor product, the total complex was obtained by tensoring together two complexes, and since we are working over a field its homology is the tensor product of the homology of the two complexes, so

$$\pi_* \left(\left(\mathcal{L}^k_{S^1} \left(k[x] \otimes \Lambda(\varepsilon x)^\tau \right) \right)_* \right) \cong \pi_* \left(\left(\mathcal{L}^k_{S^1}(k[x]) \right)_* \right) \otimes \pi_* \left(\left(\mathcal{L}^k_{S^1}(\Lambda(\varepsilon x)^\tau) \right)_* \right).$$

The first factor is just the Hochschild homology of $k[x]$. It sits in the 0th row of the E^2 term since x has internal degree zero, and gives us $\pi_*(\mathcal{L}^k_{S^1}(k[x]) \cong k[x] \otimes \Lambda(\varepsilon x))$ concentrated in positions $(0,0)$ and $(1,0)$. All spectral sequence differentials vanish on it for degree reasons, and so it will just contribute $k[x] \otimes \Lambda(\varepsilon x)$ to the E^∞ term.

The second factor in the E^2 term is the twisted Hochschild homology for $\Lambda(\varepsilon x)$. To calculate it, we can use the normalized chain complex and therefore we only have to consider non-degenerate elements, which means that we only have two elements to take into account in any given simplicial degree:

p-degree	0	1	2	...
	1	$1 \otimes \varepsilon x$	$1 \otimes \varepsilon x \otimes \varepsilon x$	...
	εx	$\varepsilon x \otimes \varepsilon x$	$\varepsilon x \otimes \varepsilon x \otimes \varepsilon x$	...

Elements of the form $\varepsilon x \otimes \ldots \otimes \varepsilon x$ will map to zero under the Hochschild boundary map. We need to consider the odd and even cases of differentials on elements of the form $1 \otimes \varepsilon x \otimes \ldots \otimes \varepsilon x$. The d_i maps in the twisted and untwisted Hochschild complex are all the same except d_0, which incorporates the twisting action of τ. Therefore we have

$$\begin{aligned} d(1 \otimes (\varepsilon x)^{\otimes 2k}) &= -\varepsilon x^{\otimes 2k} + (-1)^{2k}(-1)\varepsilon x^{\otimes 2k} = -2\varepsilon x^{\otimes 2k} \\ d(1 \otimes (\varepsilon x)^{\otimes 2k+1}) &= -\varepsilon x^{\otimes 2k+1} + (-1)^{2k+1}(1)\varepsilon x^{\otimes 2k+1} = -2\varepsilon x^{\otimes 2k+1}. \end{aligned}$$

Here, the first -1 comes from the γ action on ϵx as in (2.16) and the extra ± 1 in brackets come from passing the one-dimensional εx past an odd or an even number of copies of itself. Since we are assuming that 2 is invertible in k, we get that the second part of the E^2 term has only k left in degree 0. So, if 2 is invertible in k, then the entire E^2 term is just $k[x] \otimes \Lambda(\varepsilon x)$ in the 0th row, and the TCP spectral sequence collapses and confirms that

$$\pi_* \mathcal{L}^k_{K\ell}(k[x]) \cong k[x] \otimes \Lambda(\varepsilon x).$$

3. Hopf algebras in spectra

We start by describing what we mean by the notion of a commutative Hopf algebra in the ∞-category of spectra, Sp. We consider the ∞-category CAlg of E_∞-ring spectra.

Definition 3.1. A *commutative Hopf algebra spectrum* is a cogroup object in CAlg.

Hopf algebra spectra are fairly rare, so let us list some important examples.

Example 3.2. If G is a topological abelian group, then the spherical group ring $S[G] = \Sigma^\infty_+ G$ equipped with the product induced by the product in G, the coproduct induced by the diagonal map $G \to G \times G$, and the antipodal map induced by the inverse map from G to G is a commutative Hopf algebra spectrum. This follows from the fact that the suspension spectrum functor $\Sigma^\infty_+ \colon \mathcal{S} \to \mathrm{Sp}$ is a strong symmetric monoidal functor. Here $\mathcal{S}$ denotes the ∞-category of spaces.

Example 3.3. If A is an ordinary commutative Hopf algebra over a commutative ring k and A is flat as a k-module then the Eilenberg-Mac Lane spectrum HA is a commutative Hopf algebra spectrum over Hk because the canonical map

$$HA \wedge_{Hk} HA \to H(A \otimes_k A)$$

is an equivalence.

We use the fact that the category of commutative ring spectra is tensored over unpointed topological spaces and simplicial sets in a compatible way [8, VII, §2, §3].

If $\mathcal{U}$ denotes the category of unbased (compactly generated weak Hausdorff) spaces and $X \in \mathcal{U}$, then for every pair of commutative ring spectra A and B there is a homeomorphism of mapping spaces ([8, VII, Theorem 2.9])

$$\mathcal{C}_S(X \otimes A, B) \cong \mathcal{U}(X, \mathcal{C}_S(A, B)). \tag{3.4}$$

Here, $\mathcal{C}_S$ denotes the (ordinary) category of commutative ring spectra in the sense of [8]. By [16, Corollary 4.4.4.9], (3.4) corresponds to an equivalence of mapping spaces of ∞-categories

$$\mathrm{CAlg}(X \otimes A, B) \simeq \mathcal{S}(X, \mathrm{CAlg}(A, B)). \tag{3.5}$$

See also [22, §2] for a detailed account on tensors in ∞-categories.

If we consider a commutative Hopf algebra spectrum $\mathcal{H}$, then the space of maps $\mathcal{C}_S(\mathcal{H}, B)$ has a basepoint: the composition of the counit map $\mathcal{H} \to S$ followed by the unit map $S \to B$ is a map of commutative ring spectra. The functor that takes an unbased space X to the topological sum of X with a point $+$ is left adjoint to the forgetful functor the category of pointed spaces, Top_*, to spaces, so we obtain a homeomorphism

$$\mathcal{U}(X, \mathcal{C}_S(\mathcal{H}, B)) \cong \mathrm{Top}_*(X_+, \mathcal{C}_S(\mathcal{H}, B)) \tag{3.6}$$

and correspondingly, an equivalence in the context of ∞-categories

$$\mathcal{S}(X, \mathrm{CAlg}(\mathcal{H}, B)) \simeq \mathcal{S}_*(X_+, \mathrm{CAlg}(\mathcal{H}, B)). \tag{3.7}$$

For path-connected spaces Z, May showed that the free E_n-space on Z, $C_n(Z)$, is equivalent to $\Omega^n\Sigma^n Z$ [19, Theorem 6.1]. Segal extended this result to spaces that are not necessarily connected. He showed that for well-based spaces Z there is a model of the free E_1-space, $C_1'(Z)$, as follows: The spaces $C_1(Z)$ and $C_1'(Z)$ are homotopy equivalent, $C_1'(Z)$ is a monoid, its classifying space $BC_1'(Z)$ is equivalent to $\Sigma(Z)$ [24, Theorem 2], and thus, $C_1'(Z) \to \Omega BC_1'(Z)$ is a group completion. We can apply this result to $Z = X_+$ because X_+ is well-based, thus $BC_1'(X_+) \simeq \Sigma(X_+)$. Note that $\Omega BC_1'(X_+) \simeq \Omega\Sigma(X_+)$.

Nikolaus gives an overview about group completions in the context of ∞-categories [20]. He shows that for every E_1-monoid M, the map $M \to \Omega BM$ gives rise to a localization functor of ∞-cateories in the sense of [16, Definition 5.2.7.2], such that the local objects are grouplike E_1-spaces. In particular, there is a homotopy equivalence of mapping spaces [16, Proposition 5.2.7.4]

$$\mathrm{Map}_{E_1}(\Omega BC_1'(X_+), Y) \simeq \mathrm{Map}_{E_1}(C_1'(X_+), Y)$$

if Y is a grouplike E_1-space. Here, E_1 denotes the ∞-category of E_1-spaces.

If $\mathcal{H}$ is a commutative Hopf-algebra, then the space $\mathrm{CAlg}(\mathcal{H}, B)$ is a grouplike E_1-space. Therefore, by using Equations (3.5) and (3.7), we obtain a chain of homotopy equivalences

$$\begin{aligned} \mathrm{CAlg}(X \otimes \mathcal{H}, B) &\simeq \mathcal{S}(X, \mathrm{CAlg}(\mathcal{H}, B)) \\ &\simeq \mathcal{S}_*(X_+, \mathrm{CAlg}(\mathcal{H}, B)) \\ &\simeq \mathrm{Map}_{E_1}(C_1'(X_+), \mathrm{CAlg}(\mathcal{H}, B)) \\ &\simeq \mathrm{Map}_{E_1}(\Omega BC_1'(X_+), \mathrm{CAlg}(\mathcal{H}, B)) \\ &\simeq \mathrm{Map}_{E_1}(\Omega\Sigma(X_+), \mathrm{CAlg}(\mathcal{H}, B)). \end{aligned}$$

If $\Sigma(X_+) \simeq \Sigma(Y_+)$ is an equivalence of pointed spaces, then $\Omega\Sigma(X_+) \simeq \Omega\Sigma(Y_+)$ as grouplike E_1-spaces and therefore we get a homotopy equivalence

$$\mathrm{CAlg}(X \otimes \mathcal{H}, B) \simeq \mathrm{CAlg}(Y \otimes \mathcal{H}, B).$$

Applying the Yoneda Embedding to the above equivalence yields the following result:

Theorem 3.8. *If $\mathcal{H}$ is a commutative Hopf algebra spectrum and if $\Sigma(X_+) \simeq \Sigma(Y_+)$ is an equivalence of pointed spaces, then there is an equivalence $X \otimes \mathcal{H} \simeq Y \otimes \mathcal{H}$ in* CAlg.

Remark 3.9. If X is a pointed simplicial set, then the suspension $\Sigma(X_+)$ is equivalent to $\Sigma(X) \vee S^1$. Therefore, if X and Y are pointed simplicial sets, such that $\Sigma(X) \simeq \Sigma(Y)$ as pointed simplicial sets, then we also obtain an equivalence between $\Sigma(X_+)$ and $\Sigma(Y_+)$.

Segal's result also works for larger n than 1. If two spaces are equivalent after an n-fold suspension, then an E_n-coalgebra structure on a Hopf algebra is needed for the Loday construction to be equivalent on these two spaces. There are indeed interesting spaces that are not equivalent after just one suspension, but that need iterated suspensions to become equivalent:

- Christoph Schaper [23, Theorem 3] shows that for affine arrangements $\mathcal{A}$ one needs at least a $(\tau_{\mathcal{A}}+2)$-fold suspension in order to get a homotopy type that only depends on the poset structure of the arrangement. Here, $\tau_{\mathcal{A}}$ is a number that depends on the poset data of the arrangement, namely the intersection poset and the dimension function.
- For homology spheres, the double suspension theorem of James W. Cannon and Robert D. Edwards [6, Theorem in §11] states that the double suspension $\Sigma^2 M$ of any n-dimensional homology sphere M is homeomorphic to S^{n+2}. Here, a single suspension does *not* suffice unless M is an actual sphere.

4. Truncated polynomial algebras

One way of showing that a commutative R-algebra spectrum A is not multiplicatively or linearly stable is to prove that the homotopy groups of the Loday construction $\mathcal{L}^R_{T^n}(A)$ differ from those of $\mathcal{L}^R_{\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} S^k}(A)$, as in [7]. Here, we write $\bigvee_{\binom{n}{k}} S^k$ for the $\binom{n}{k}$-fold $\vee$-sum of S^k. Indeed, there is a homotopy equivalence

$$\Sigma(T^n) \simeq \Sigma(\bigvee_{k=1}^{n} \bigvee_{\binom{n}{k}} S^k).$$

If A is augmented over R, then for proving that $R \to A$ is not multiplicatively or additively stable, it suffices to show that

$$\mathcal{L}^R_{T^n}(A;R) \not\simeq \mathcal{L}^R_{\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} S^k}(A;R).$$

See [14, §2] for details and background on different notions of stability.

In the following we restrict our attention to Eilenberg-Mac Lane spectra of commutative rings and we will use this strategy to show that none of the commutative $\mathbb{Q}$-algebras $\mathbb{Q}[t]/t^m$ for $m \geqslant 2$ can be multiplicatively stable. We later generalize this to quotients of the form $\mathbb{Q}[t]/q(t)$ where $q(t)$ is a polynomial without constant term and to integral and mod-p results.

Pirashvili determined higher order Hochschild homology of truncated polynomial algebras of the form $k[x]/x^{r+1}$ additively when k is a field of characteristic zero [21, Section 5.4] in the case of odd spheres. A direct adaptation of the methods of [4, Theorem 8.8] together with the flowchart from [5, Proposition 2.1] yields the higher order Hochschild homology with reduced coefficients for all spheres. See also [7, Lemma 3.4].

Proposition 4.1. *For all $m \geqslant 2$ and $n \geqslant 1$*

$$\mathsf{HH}^{[n],\mathbb{Q}}_*(\mathbb{Q}[t]/t^m;\mathbb{Q}) \cong \begin{cases} \Lambda_{\mathbb{Q}}(x_n) \otimes \mathbb{Q}[y_{n+1}], & \text{if } n \text{ is odd,} \\ \mathbb{Q}[x_n] \otimes \Lambda_{\mathbb{Q}}(y_{n+1}), & \text{if } n \text{ is even.} \end{cases}$$

In both cases Hochschild homology of order n is a free graded commutative $\mathbb{Q}$-algebra on two generators in degrees n and $n+1$, respectively, and the result does not depend on m.

We will determine for which m and n we get a decomposition of the form

$$\pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t]/t^m;\mathbb{Q}) \cong \pi_*\mathcal{L}^{\mathbb{Q}}_{\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} S^k}(\mathbb{Q}[t]/t^m;\mathbb{Q}). \tag{4.2}$$

Note that the right-hand side is isomorphic to

$$\bigotimes_{k=1}^{n} \bigotimes_{\binom{n}{k}} \pi_*\mathcal{L}^{\mathbb{Q}}_{S^k}(\mathbb{Q}[t]/t^m;\mathbb{Q})$$

where all unadorned tensor products are formed over $\mathbb{Q}$. Thus, if we have a decomposition as in (4.2), then we can read off the homotopy groups of $\pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t]/t^m;\mathbb{Q})$ with the help of Proposition 4.1.

Expressing $\mathbb{Q}[t]/t^m$ as the pushout of the diagram

$$\begin{tikzcd}
\mathbb{Q}[t] \arrow[r, "t\mapsto t^m"] \arrow[d, "t\mapsto 0"'] & \mathbb{Q}[t] \\
\mathbb{Q} &
\end{tikzcd}$$

allows us to express the Loday construction for $\mathbb{Q}[t]/t^m$, now viewed as a commutative $H\mathbb{Q}$-algebra spectrum, as the homotopy pushout of the diagram

$$\begin{tikzcd}
\mathcal{L}^{H\mathbb{Q}}_{T^n}(H\mathbb{Q}[t];H\mathbb{Q}) \arrow[r, "t\mapsto t^m"] \arrow[d, "t\mapsto 0"'] & \mathcal{L}^{H\mathbb{Q}}_{T^n}(H\mathbb{Q}[t];H\mathbb{Q}) \\
H\mathbb{Q} &
\end{tikzcd}$$

and so

$$\mathcal{L}^{H\mathbb{Q}}_{T^n}(H\mathbb{Q}[t]/t^m;H\mathbb{Q}) \simeq \mathcal{L}^{H\mathbb{Q}}_{T^n}(H\mathbb{Q}[t];H\mathbb{Q}) \wedge^L_{\mathcal{L}^{H\mathbb{Q}}_{T^n}(H\mathbb{Q}[t];H\mathbb{Q})} H\mathbb{Q}.$$

As $\mathbb{Q}[t]$ is smooth over $\mathbb{Q}$, $\mathcal{L}^{H\mathbb{Q}}_{T^n}(H\mathbb{Q}[t];H\mathbb{Q})$ is stable [7, Example 2.6]. So we can write

$$\mathcal{L}^{H\mathbb{Q}}_{T^n}(H\mathbb{Q}[t];H\mathbb{Q}) \simeq \mathcal{L}^{H\mathbb{Q}}_{\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} S^k}(H\mathbb{Q}[t];H\mathbb{Q}).$$

Again, we obtain an isomorphism

$$\pi_*\mathcal{L}^{\mathbb{Q}}_{\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} S^k}(\mathbb{Q}[t];\mathbb{Q}) \cong \bigotimes_{k=1}^{n} \bigotimes_{\binom{n}{k}} \mathsf{HH}^{[k],\mathbb{Q}}_*(\mathbb{Q}[t];\mathbb{Q})$$

and with the help of [5, Proposition 2.1] we can identify the terms as follows:

$$\mathsf{HH}^{[k],\mathbb{Q}}_*(\mathbb{Q}[t];\mathbb{Q}) \cong \begin{cases} \mathbb{Q}[x_k], & \text{if } k \text{ is even },\\ \Lambda_{\mathbb{Q}}(x_k), & \text{if } k \text{ is odd.}\end{cases}$$

Lemma 4.3. *There is an isomorphism of graded commutative $\mathbb{Q}$-algebras*

$$\pi_*\mathcal{L}^{\mathbb{Q}}_{\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} S^k}(\mathbb{Q}[t]/t^m;\mathbb{Q}) \cong \pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q}) \otimes \mathrm{Tor}_*^{\pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})}(\mathbb{Q},\mathbb{Q}).$$

Proof. We already know that

$$\pi_*\mathcal{L}^{\mathbb{Q}}_{\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} S^k}(\mathbb{Q}[t]/t^m;\mathbb{Q}) \cong \bigotimes_{k=1}^{n} \bigotimes_{\binom{n}{k}} \mathsf{HH}^{[k],\mathbb{Q}}_*(\mathbb{Q}[t]/t^m;\mathbb{Q}) \cong \bigotimes_{k=1}^{n} \bigotimes_{\binom{n}{k}} gF_{\mathbb{Q}}(x_k, y_{k+1}) \tag{4.4}$$

where $gF_{\mathbb{Q}}(x_k)$ denotes the free graded commutative $\mathbb{Q}$-algebra generated by an element x_k in degree k and $gF_{\mathbb{Q}}(x_k, y_{k+1})$ denotes the free graded commutative $\mathbb{Q}$-algebra generated by an element x_k in degree k and an element y_{k+1} in degree $k+1$.

As $\pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q}) \cong \bigotimes_{k=1}^n \bigotimes_{\binom{n}{k}} gF_{\mathbb{Q}}(x_k)$, we obtain that

$$\mathrm{Tor}_*^{\pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})}(\mathbb{Q},\mathbb{Q}) \cong \bigotimes_{\ell=2}^{n+1} \bigotimes_{\binom{n}{\ell-1}} gF_{\mathbb{Q}}(y_\ell)$$

and hence the tensor product of the two gives a graded commutative $\mathbb{Q}$-algebra isomorphic to (4.4) □

Let A_* denote the graded commutative $\mathbb{Q}$-algebra $\pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})$ and B_* denote $\pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})$ viewed as an A_*-module via a morphism of graded commutative $\mathbb{Q}$-algebras $f\colon A_* \to B_*$.

Lemma 4.5. *Let $f_1\colon A_* \to B_*$ be the morphism $f_1 = \eta_{B_*} \circ \varepsilon_{A_*}$ where $\varepsilon_{A_*}\colon A_* \to \mathbb{Q}$ is the augmentation that sends all elements of positive degree to zero and where $\eta_{B_*}\colon \mathbb{Q} \to B_*$ is the unit map of B_*. Let $f_2\colon A_* \to B_*$ be any map of graded commutative algebras such that there is an element $x \in A_n$ with $n > 0$ such that $f_2(x) = w \neq 0$. Let $\mathrm{Tor}^{A_*,f_i}_{*,*}(B_*,\mathbb{Q})$ denote the graded* Tor*-groups calculated with respect to the A_*-module structure on B_* given by f_i. Then*

$$\dim_{\mathbb{Q}}(\mathrm{Tor}^{A_*,f_2}_{*,*}(B_*,\mathbb{Q}))_n < \dim_{\mathbb{Q}}(\mathrm{Tor}^{A_*,f_1}_{*,*}(B_*,\mathbb{Q}))_n$$

where $(\mathrm{Tor}^{A_*,f_i}_{*,*}(B_*,\mathbb{Q}))_n = \bigoplus_{r+s=n} \mathrm{Tor}^{A_*,f_i}_{r,s}(B_*,\mathbb{Q})$.

Proof. Let $P_* \to \mathbb{Q}$ be an A_*-free resolution of $\mathbb{Q}$. We want to choose P_* efficiently, in the following sense: since $\mathbb{Q}$ is concentrated in degree zero and $A_0 = \mathbb{Q}$, we can choose P_0 to be A_*. Then we choose $P_1 = \bigoplus_{j\in I_1} \Sigma^{n_j} A_*$ with the minimal possible number of copies of A_* in each suspension degree, beginning from the bottom (that is, the only reason we add a new $\Sigma^n A_*$ is if there is a class in $P_{\ell-1}$ that has not yet been hit by the suspensions of A_* in lower dimensions that we already have) to guarantee that $d_1\colon \bigoplus_{j\in I_\ell} \Sigma^{n_j} A_0 \to P_0$ is injective, and moreover

$$\ker(d_1\colon \bigoplus_{j\in I_1} \Sigma^{n_j} A_0 \to P_0) \subseteq \bigoplus_{j\in I_1} \Sigma^{n_j} \ker(\epsilon_{A_*}).$$

And of course we need $\mathrm{Im}(d_1\colon P_1 \to P_0) = \ker(\epsilon_{A_*}\colon A_* \to \mathbb{Q})$ and similarly for higher ℓ. For every $\ell > 0$ we choose P_ℓ with

$$P_\ell = \bigoplus_{j\in I_\ell} \Sigma^{n_j} A_*$$

so that $d_\ell\colon \bigoplus_{j\in I_\ell} \Sigma^{n_j} A_0 \to P_{\ell-1}$ is injective and moreover

$$\ker(d_\ell\colon \bigoplus_{j\in I_\ell} \Sigma^{n_j} A_0 \to P_{\ell-1}) \subseteq \bigoplus_{j\in I_\ell} \Sigma^{n_j} \ker(\epsilon_{A_*}).$$

Then we get

$$\mathrm{Im}(d_\ell\colon P_\ell \to P_{\ell-1}) = \ker(d_{\ell-1}\colon P_{\ell-1} \to P_{\ell-2}) \subseteq \bigoplus_{j\in I_{\ell-1}} \Sigma^{n_j} \ker(\epsilon_{A_*}).$$

The Tor groups we want are the homology groups of

$$B_* \otimes_{A_*} P_\ell = B_* \otimes_{A_*} \bigoplus_{j\in I_\ell} \Sigma^{n_j} A_* \cong \bigoplus_{j\in I_\ell} \Sigma^{n_j} B_*$$

with respect to the differential $\mathrm{id} \otimes d$ for either A_*-module structure.

As $f_1\colon A_* \to B_*$ factors through the augmentation, we claim that the differentials in the chain complex

$$B_* \otimes_{A_*} P_\bullet$$

with the A_*-module structure given by f_1 are trivial: they are of the form $\mathrm{id} \otimes d$ where d is the differential of $P_\bullet$. As d sends every $\Sigma^{n_j} 1 \in \bigoplus_{j\in I_\ell} \Sigma^{n_j} A_*$ to something in $\bigoplus_{j\in I_{\ell-1}} \Sigma^{n_j} \ker(\epsilon_{A_*})$,

$$(\mathrm{id} \otimes d)(b \otimes_{A_*} \Sigma^{n_j} 1) \in \mathbb{Q}\{b\} \otimes_{A_*} \bigoplus_{j\in I_{\ell-1}} \Sigma^{n_j} \ker(\epsilon_{A_*}) = 0$$

for all $b \in B_*$. Hence

$$\mathrm{Tor}^{A_*,f_1}_{\ell,s}(B_*,\mathbb{Q}) = (\bigoplus_{j\in I_\ell} \Sigma^{n_j} B_*)_s = \bigoplus_{j\in I_\ell} \Sigma^{n_j} B_{*-s}.$$

In particular, we have of course $\mathrm{Tor}_{0,s}^{A_*,f_1}(B_*,\mathbb{Q}) = B_s$ for all s.

For the A_*-module structure on B_* given by f_2 we obtain that

$$\mathrm{Tor}_{0,*}^{A_*,f_2}(B_*,\mathbb{Q}) = B_* \otimes_{A_*} \mathbb{Q}$$

but here, the tensor product results in a nontrivial quotient of B_*. Recall that we assumed that $f_2(x) = w \neq 0$. The element $w \otimes 1 \in B_* \otimes_{A_*} \mathbb{Q}$ is trivial because the degree of x is positive and hence $\varepsilon_{A_*}(x) = 0$:

$$w \otimes 1 = f_2(x) \otimes 1 = 1 \otimes \varepsilon_{A_*}(x) = 1 \otimes 0 = 0.$$

Therefore,

$$\dim_{\mathbb{Q}} \mathrm{Tor}_{0,n}^{A_*,f_2}(B_*,\mathbb{Q}) < \dim_{\mathbb{Q}} \mathrm{Tor}_{0,n}^{A_*,f_1}(B_*,\mathbb{Q}).$$

The other Tor-terms in total degree n of the form $\mathrm{Tor}_{r,s}^{A_*,f_2}(B_*,\mathbb{Q})$ with $r+s=n$ are subquotients of

$$\bigoplus_{j \in I_r} \Sigma^{n_j} B_{*-s}$$

and hence for all (r,s) with $r+s=n$ and $r>0$ we obtain

$$\dim_{\mathbb{Q}} \mathrm{Tor}_{r,s}^{A_*,f_2}(B_*,\mathbb{Q}) \leqslant \dim_{\mathbb{Q}} \mathrm{Tor}_{r,s}^{A_*,f_1}(B_*,\mathbb{Q}).$$

□

Note that if $f\colon A_* \to B_*$ factors through the augmentation $A_* \to \mathbb{Q}$ then

$$\mathrm{Tor}_{\ell}^{A_*}(B_*,\mathbb{Q}) \cong B_* \otimes \mathrm{Tor}_{\ell}^{A_*}(\mathbb{Q},\mathbb{Q}).$$

We use Lemma 4.5 to prove the following result.

Theorem 4.6. *Let $n \geqslant 2$. Then*

$$\dim_{\mathbb{Q}} \pi_n \mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t]/t^n;\mathbb{Q}) < \dim_{\mathbb{Q}} \pi_n \mathcal{L}^{\mathbb{Q}}_{\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} S^k}(\mathbb{Q}[t]/t^n;\mathbb{Q}).$$

In particular, for all $n \geqslant 2$ the pair $(\mathbb{Q}[t]/t^n;\mathbb{Q})$ is not stable and $\mathbb{Q} \to \mathbb{Q}[t]/t^n$ is not multiplicatively stable.

The $n=2$ case of Theorem 4.6 was obtained earlier by Dundas and Tenti [7].

Before we prove the theorem, we state the following integral version of it:

Corollary 4.7. *For all $n \geqslant 2$ the pair $(\mathbb{Z}[t]/t^n;\mathbb{Z})$ is not stable and $\mathbb{Z} \to \mathbb{Z}[t]/t^n$ is not multiplicatively stable.*

Proof of Corollary 4.7. If for some $n \geqslant 2$ the pair $(\mathbb{Z}[t]/t^n;\mathbb{Z})$ were stable, then in particular

$$\pi_* \mathcal{L}^{\mathbb{Z}}_{T^n}(\mathbb{Z}[t]/t^n;\mathbb{Z}) \cong \pi_* \mathcal{L}^{\mathbb{Z}}_{\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} S^k}(\mathbb{Z}[t]/t^n;\mathbb{Z}).$$

Localizing at $\mathbb{Z} \setminus \{0\}$ would then imply

$$\pi_* \mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t]/t^n;\mathbb{Q}) \cong \pi_* \mathcal{L}^{\mathbb{Q}}_{\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} S^k}(\mathbb{Q}[t]/t^n;\mathbb{Q})$$

in contradiction to Theorem 4.6. □

4.1. **Proof of Theorem 4.6.** We prove Theorem 4.6 by identifying an element in A_* of positive degree that is sent to a nontrivial element of B_*. More precisely, we will show that the map that sends t to t^n sends the indecomposable element in $\pi_n\mathcal{L}^{\mathbb{Q}}_{S^n}(\mathbb{Q}[t];\mathbb{Q})$ up to a unit to the element

$$dt_1 \wedge \ldots \wedge dt_n \in \pi_n\mathcal{L}^{\mathbb{Q}}_{S^1\vee\ldots\vee S^1}(\mathbb{Q}[t];\mathbb{Q}).$$

We consider both elements as elements of $\pi_n\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})$ via the inclusions of summands

$$\pi_n\mathcal{L}^{\mathbb{Q}}_{S^1\vee\ldots\vee S^1}(\mathbb{Q}[t];\mathbb{Q}) \subset \pi_n\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q}) \supset \pi_n\mathcal{L}^{\mathbb{Q}}_{S^n}(\mathbb{Q}[t];\mathbb{Q}).$$

In the following we consider T^n as the diagonal of an n-fold simplicial set where every $([p_1],\ldots,[p_n]) \in (\Delta)^n$ is mapped to $S^1_{p_1} \times \ldots \times S^1_{p_n}$. Then $\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})$ can also be interpreted as the diagonal of an n-fold simplicial $\mathbb{Q}$-vector space with an associated n-chain complex. By abuse of notation we still denote this n-chain complex by $\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})$.

We use the following notation concerning the n-chain complex $\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})$:

- $\mathbf{0}_m = (0,0,\ldots,0)$ and $\mathbf{1}_m = (1,1,\ldots,1)$ are the vectors containing only 0 or 1, respectively, repeated m times.
- A vector $\mathbf{V} \in \mathbb{N}^n$ is viewed as a *multi-degree* of an element in the n-chain complex.
- A vector $\mathbf{v} \in \mathbb{N}^n$ for which $\mathbf{0}_n \leqslant \mathbf{v} \leqslant \mathbf{V}$ in every entry can be thought of as specifying a *coordinate* in the multi-matrix of an element in multi-degree $\mathbf{V}$. We call the ith entry of a vector $\mathbf{v} \in \mathbb{N}^n$ the ith *place* in $\mathbf{v}$. It is always assumed that $\mathbf{V} = \mathbf{1}_n$ if not otherwise specified.
- Each element of $\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})$ in degree $\mathbf{V} = (v_1,\ldots,v_n)$ is a multi-matrix of dimension $(v_1+1,\ldots,v_n+1)$ with entries in $\mathbb{Q}[t]$ at coordinates $\mathbf{v} \neq \mathbf{0}_n$ and an entry in $\mathbb{Q}$ at coordinate $\mathbf{0}_n$.
- $x_{\mathbf{v}}$ for $x \in \mathbb{Q}[t]$ and $\mathbf{v} \in \mathbb{N}^n$ is the multi-matrix with term x at coordinate $\mathbf{v}$ and 1 at other coordinates. We say a term is trivial if it is 1 in all its coordinates.
- Therefore $x_{\mathbf{v}} \cdot y_{\mathbf{w}}$ for $x, y \in \mathbb{Q}[t]$ and $\mathbf{v}, \mathbf{w} \in \mathbb{N}^n$ is the product of $x_{\mathbf{v}}$ and $y_{\mathbf{w}}$ in degree $\mathbf{V}$ of $\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t],\mathbb{Q})$ regarded as an n-simplicial ring. Explicitly, if $\mathbf{v} \neq \mathbf{w}$, it is the multi-matrix with x at coordinate $\mathbf{v}$, y at coordinate $\mathbf{w}$, and 1 elsewhere; if $\mathbf{v} = \mathbf{w}$, it is the multi-matrix with xy at coordiante $\mathbf{v}$ and 1 elsewhere.

Suppose that $C_\bullet$ is an n-chain complex with differentials $\mathrm{d}_1,\ldots,\mathrm{d}_n$ in the n different directions, then the total chain complex $\mathrm{Tot}(C_\bullet)$ has differential in component $(v_1,\ldots,v_n)$ given by

$$\mathrm{d} = \sum_{i=1}^{n} (-1)^{v_1+\ldots+v_{i-1}} \mathrm{d}_i.$$

In our case we will have each $\mathrm{d}_i = \sum_{j=0}^{v_i} (-1)^j d_{i,j}$ where $d_{i,j}\colon C_{v_1,\ldots,v_n} \to C_{v_1,\ldots,v_i-1,\ldots v_n}$ is the face map. We are interested in low degrees, especially in $\mathbf{1}_n$. Any $v_i = 1$ will imply $\mathrm{d}_i = 0$ since the d_i are cyclic differentials and $\mathbb{Q}[t]$ is commutative. This allows us to eliminate the d_i from d. We have the following three lemmas about homologous classes and tori of different dimensions:

Lemma 4.8 (Split Moving Lemma)**.** *Let* $\mathbf{a}, \mathbf{b}$ *be coordinates in degree* $\mathbf{1}_{n-1}$ *(that is, in* $2\times 2\times\ldots\times 2$*-dimensional matrices). Then*

$$x_{(\mathbf{a},1)} \cdot y_{(\mathbf{b},1)} \sim x_{(\mathbf{a},0)} \cdot y_{(\mathbf{b},1)} + x_{(\mathbf{a},1)} \cdot y_{(\mathbf{b},0)}.$$

Proof. Their difference is a boundary of an element of degree $(\mathbf{1}_{n-1}, 2)$:

$$\mathrm{d}(x_{(\mathbf{a},1)} \cdot y_{(\mathbf{b},2)}) = (-1)^{n-1}\mathrm{d}_n(x_{(\mathbf{a},1)} \cdot y_{(\mathbf{b},2)}) = x_{(\mathbf{a},0)} \cdot y_{(\mathbf{b},1)} - x_{(\mathbf{a},1)} \cdot y_{(\mathbf{b},1)} + x_{(\mathbf{a},1)} \cdot y_{(\mathbf{b},0)}.$$

□

For example, when $n=2$, $\mathbf{a}=0, \mathbf{b}=1$, the difference is

$$\mathrm{d}\begin{pmatrix}1 & x & 1\\ 1 & 1 & y\end{pmatrix} = \begin{pmatrix}x & 1\\ 1 & y\end{pmatrix} - \begin{pmatrix}1 & x\\ 1 & y\end{pmatrix} + \begin{pmatrix}1 & x\\ y & 1\end{pmatrix}.$$

Let $\mathbf{b}$ be a coordinate of a multi-matrix of an element in degree $\mathbf{1}_{n-m}$ such that $\mathbf{b} \neq \mathbf{0}_{n-m}$. For any multi-matrix c in degree $\mathbf{W} \in \mathbb{N}^m$, we can form the following multi-matrix in degree $(\mathbf{W}, \mathbf{1}_{n-m}) \in \mathbb{N}^n$:

$$c_{(-,\mathbf{0})} \otimes y_{(\mathbf{0},\mathbf{b})} \text{ has terms } \begin{cases} c_{\mathbf{a}} & \text{at coordinate } (\mathbf{a}, \mathbf{0}_{n-m});\\ y_{\mathbf{b}} & \text{at coordinate } (\mathbf{0}_m, \mathbf{b});\\ 1 & \text{elsewhere.}\end{cases}$$

Lemma 4.9. *The following is a chain map:*

$$\begin{array}{ccc} \mathrm{Tot}(\mathcal{L}^{\mathbb{Q}}_{T^m}(\mathbb{Q}[t],\mathbb{Q})) & \to & \mathrm{Tot}(\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t],\mathbb{Q}));\\ c & \mapsto & c_{(-,\mathbf{0})} \otimes y_{(\mathbf{0},\mathbf{b})}.\end{array}$$

Proof. Clearly $\mathrm{d}_i(c_{(-,\mathbf{0})} \otimes y_{(\mathbf{0},\mathbf{b})}) = \mathrm{d}_i c_{(-,\mathbf{0})} \otimes y_{(\mathbf{0},\mathbf{b})}$ for $0 \leqslant i \leqslant m$. But since the multi-degree of $c_{(-,\mathbf{0})} \otimes y_{(\mathbf{0},\mathbf{b})}$ is $\mathbf{V} = (\mathbf{W}, \mathbf{1}_{n-m}) \in \mathbb{N}^n$ and whenever $v_i = 1$, $\mathrm{d}_i = d_{i,0} - d_{i,1} = 0$, we also get

$$\mathrm{d}_i(c_{(-,\mathbf{0})} \otimes y_{(\mathbf{0},\mathbf{b})}) = 0, \text{ for } m < i \leqslant n.$$

$\square$

This lemma also applies when $y_{(\mathbf{0},\mathbf{b})}$ is replaced by another multi-matrix that has more than one nontrivial term, as long as the nontrivial terms are all in coordinates of the form $(\mathbf{0}_m, \mathbf{b})$ for $\mathbf{b}$ in degree $\mathbf{1}_{n-m}$ and $\mathbf{b} \neq \mathbf{0}_{n-m}$. It has the following immediate corollary:

Lemma 4.10 (Orthogonal Moving Lemma)**.** *Let $\mathbf{b}$ be a coordinate in degree $\mathbf{1}_{n-m}$ such that $\mathbf{b} \neq \mathbf{0}_{n-m}$. Let c, c' be elements in multi-degree $\mathbf{W} \in \mathbb{N}^m$. If $c \sim c'$ in multi-degree $\mathbf{W}$, then*

$$c_{(-,\mathbf{0}_{n-m})} \otimes y_{(\mathbf{0}_m,\mathbf{b})} \sim c'_{(-,\mathbf{0}_{n-m})} \otimes y_{(\mathbf{0}_m,\mathbf{b})}$$

in multi-degree $(\mathbf{W}, \mathbf{1}_{n-m})$

Conceptually, the moving lemmas tell us how to move the nontrivial elements x, y in certain multi-matrices to lower coordinates. They are stated for a special case for simplicity, but of course they work for any permulation of copies of $\mathbb{N}^n$ in the statement. The split moving lemma says that if we have $x_{\mathbf{v}}$ and $y_{\mathbf{w}}$ where the coordinates share a 1 in a particular place, the 1's can be moved to coordinate 0 separately. The orthogonal moving lemma says that the x in $x_{\mathbf{v}}$ and the y in $y_{\mathbf{w}}$ can be moved separately if they are supported in orthogonal tori (that is, have their nontrivial entries in different coordinates).

Proposition 4.11. *Let $\mathbf{v}$ and $\mathbf{w}$ be two coordinates of degree $\mathbf{1}_n$.*

(1) If $\mathbf{v}$ and $\mathbf{w}$ are both 0 in the ith place for some $1 \leqslant i \leqslant n$, then

$$x_{\mathbf{v}} \cdot y_{\mathbf{w}} \sim 0.$$

In particular, if $\mathbf{v} \neq \mathbf{1}_n$, then $x_{\mathbf{v}} \sim 0$.

(2) In general,

$$x_{\mathbf{v}} \cdot y_{\mathbf{w}} \sim \sum_{\substack{\mathbf{v}' \leqslant \mathbf{v}, \mathbf{w}' \leqslant \mathbf{w},\\ \mathbf{v}' + \mathbf{w}' = \mathbf{1}_n}} x_{\mathbf{v}'} \cdot y_{\mathbf{w}'},$$

where the sum is taken over all coordinates $\mathbf{v}'$ and $\mathbf{w}'$ such that

- *They are place-wise no greater than $\mathbf{v}$ and $\mathbf{w}$ respectively;*
- *They take 1 in complementary places.*

(3) For $k \geqslant 1$ and $n \geqslant 1$, we have the following homologous relation:

$$(t^k)_{\mathbf{1}_n} \sim \sum_{\substack{\mathbf{w}_1,\ldots,\mathbf{w}_k \neq \mathbf{0}_n, \\ \mathbf{w}_1+\ldots+\mathbf{w}_k=\mathbf{1}_n}} \prod_{i=1}^{k} t_{\mathbf{w}_i}$$

In particular, if $k = n$ and we let $\mathbf{e}_i$ denote the coordinate that has 1 at the ith place and 0 at other places, we get

$$(t^n)_{\mathbf{1}_n} \sim n! \prod_{i=1}^{n} t_{\mathbf{e}_i}. \tag{4.12}$$

Also, if $k > n$, this gives us

$$(t^k)_{\mathbf{1}_n} \sim 0$$

Proof. The class in *(1)* is a cycle because everything is in multi-degree $\mathbf{1}_n$ is a cycle; it is null-homologous because it is in the image of the degeneracy $s_{i,0}$ in the ith place.

For *(2)* we write $|\mathbf{v}|$ for the sum of the places of the vector $\mathbf{v}$. We induct on $|\mathbf{v}| + |\mathbf{w}|$. Notice that a coordinate $\mathbf{v}$ of degree $\mathbf{1}_n$ is just a sequence of length n of 0's and 1's and $|\mathbf{v}|$ is just the number of 1's in it.

For $|\mathbf{v}| + |\mathbf{w}| \leqslant n$, there are two cases: One is that $\mathbf{v}$ and $\mathbf{w}$ are both 0 in one place. Then the claim holds because the right-hand side is the empty sum and the left-hand side is 0 by part *(1)*. The other case is that $\mathbf{v} + \mathbf{w} = \mathbf{1}_n$. Then the claim also holds because the right-hand side has only one copy that is exactly the left-hand side.

Assume that the claim is true for $|\mathbf{v}|+|\mathbf{w}| \leqslant m$ where $m \geqslant n$ and suppose now $|\mathbf{v}|+|\mathbf{w}| = m+1$. Since $m + 1 \geqslant n + 1$, $\mathbf{v}$ and $\mathbf{w}$ have to be both 1 in some place. Without loss of generality, we assume that

$$\mathbf{v} = (\mathbf{v_0}, 1),\ \ \mathbf{w} = (\mathbf{w_0}, 1) \text{ where } \mathbf{v_0}, \mathbf{w_0} \leqslant \mathbf{1}_{n-1}.$$

By the Split Moving Lemma (Lemma 4.8),

$$x_{\mathbf{v}} \cdot y_{\mathbf{w}} \sim x_{(\mathbf{v_0},0)} \cdot y_{\mathbf{w}} + x_{\mathbf{v}} \cdot y_{(\mathbf{w_0},0)}.$$

Since $|(\mathbf{v_0}, 0)| + |\mathbf{w}| = |\mathbf{v}| + |(\mathbf{w_0}, 0)| = m$, by inductive hypothesis we have that

$$\begin{aligned} x_{\mathbf{v}} \cdot y_{\mathbf{w}} &\sim \sum_{\substack{\mathbf{v_0}' \leqslant \mathbf{v_0}, \mathbf{w}' \leqslant \mathbf{w}, \\ (\mathbf{v_0}',0)+\mathbf{w}'=\mathbf{1}_n}} x_{(\mathbf{v_0}',0)} \cdot y_{\mathbf{w}'} + \sum_{\substack{\mathbf{v}' \leqslant \mathbf{v}, \mathbf{w_0}' \leqslant \mathbf{w_0}, \\ \mathbf{v}'+(\mathbf{w_0}',0)=\mathbf{1}_n}} x_{\mathbf{v}'} \cdot y_{(\mathbf{w_0}',0)} \\ &= \sum_{\substack{\mathbf{v}' \leqslant \mathbf{v}, \mathbf{w}' \leqslant \mathbf{w}, \\ \mathbf{v}'+\mathbf{w}'=\mathbf{1}_n}} x_{\mathbf{v}'} \cdot y_{\mathbf{w}'}. \end{aligned}$$

For *(3)* we order the pair (k, n) by the lexicographical ordering. We induct on (k, n). When $k = 1$, the claim is trivially true.

Suppose the claim is true for all pairs less than (k, n) where $k \geqslant 2$. Taking $\mathbf{v} = \mathbf{w} = \mathbf{1}_n$, $x = t$ and $y = t^{k-1}$ in part *(2)*, we get that

$$(t^k)_{\mathbf{1}_n} \sim \sum_{\mathbf{w}_1+\mathbf{v}'=\mathbf{1}_n} t_{\mathbf{w}_1} \cdot (t^{k-1})_{\mathbf{v}'} = \sum_{\substack{\mathbf{w}_1 \neq \mathbf{0}_n \\ \mathbf{w}_1+\mathbf{v}'=\mathbf{1}_n}} t_{\mathbf{w}_1} \cdot (t^{k-1})_{\mathbf{v}'}. \tag{4.13}$$

The second step above uses that $t_{\mathbf{0}_n} = 0$ because t is 0 in the $\mathbb{Q}[t]$-module $\mathbb{Q}$. Let $m = |\mathbf{v}'|$. By the inductive hypothesis, we have

$$(t^{k-1})_{\mathbf{1}_m} \sim \sum_{\substack{\mathbf{w}'_2,\ldots,\mathbf{w}'_k \neq \mathbf{0}_m,\\ \mathbf{w}'_2+\ldots+\mathbf{w}'_k=\mathbf{1}_m}} \prod_{i=2}^{k} t_{\mathbf{w}'_i} \tag{4.14}$$

For each $\mathbf{w}'_i$ which is a coordinate of degree $\mathbf{1}_m$, we add in 0 in places where $\mathbf{v}'$ is 0 to make it a coordinate of degree $\mathbf{1}_n$. Denote it by $\mathbf{w}_i$. Then the Orthogonal Moving Lemma (Lemma 4.10), (4.13) and (4.14) combine to

$$(t^k)_{\mathbf{1}_n} \sim \sum_{\substack{\mathbf{w}_1,\ldots,\mathbf{w}_k \neq \mathbf{0}_n,\\ \mathbf{w}_1+\ldots+\mathbf{w}_k=\mathbf{1}_n}} \prod_{i=1}^{k} t_{\mathbf{w}_i}.$$

□

For any $n \geqslant 2$, we call $t_{\mathbf{1}_n}$ the *diagonal class* and denote it by Δ_n. We call $\prod_{i=1}^{n} t_{\mathbf{e}_i}$ the *volume form* and denote it by vol_n. If we include $S^1 \hookrightarrow T^n$ as the ith coordinate and identify the first Hochschild homology group with the Kähler differentials, the generator dt of $\mathsf{HH}_1^{\mathbb{Q}}(\mathbb{Q}[t];\mathbb{Q})$ maps to the generator we call dt_i in the Loday construction of the torus. In this sense vol_n corresponds to the degree-n class $dt_1 \wedge \ldots \wedge dt_n$.

Proof of Theorem 4.6. By Equation (4.12) we know that the map $t \mapsto t^n$ induces a map on $\pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})$, that sends the diagonal class, Δ_n, to $n!\mathsf{vol}_n$. Hence, by Lemma 4.5 we know that

$$\dim_{\mathbb{Q}} \pi_n(\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t]/t^n;\mathbb{Q})) < \dim_{\mathbb{Q}} \pi_n(\mathcal{L}^{\mathbb{Q}}_{\bigvee_{k=1}^{n}\bigvee_{\binom{n}{k}} S^k}(\mathbb{Q}[t]/t^n;\mathbb{Q})).$$

In particular,

$$\pi_*(\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t]/t^n;\mathbb{Q})) \ncong \pi_*(\mathcal{L}^{\mathbb{Q}}_{\bigvee_{k=1}^{n}\bigvee_{\binom{n}{k}} S^k}(\mathbb{Q}[t]/t^n;\mathbb{Q})).$$

□

Remark 4.15. For the non-reduced Loday construction $\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t])$, parts *(1)* and *(2)* of Proposition 4.11 are still true. Part *(3)* will become

$$(t^k)_{\mathbf{1}_n} \sim \sum_{\mathbf{w}_1+\ldots+\mathbf{w}_k=\mathbf{1}_n} \prod_{i=1}^{k} t_{\mathbf{w}_i}$$

and Equation (4.12) is no longer true.

4.2. $\mathbb{Q}[t]/t^m$ **on** T^n **for** $2 \leqslant m < n$**.** We know that for $\mathbb{Q}[t]/t^n$ we get a discrepancy between π_n of the Loday construction on the n-torus and that of the bouquet of spheres that correspond to the cells of the n-torus. We use this to first show that $\mathbb{Q}[t]/t^m$ causes a similar discrepancy for $2 \leqslant m < n$.

Proposition 4.16. *Let $2 \leqslant m \leqslant n$. Then*

$$\pi_m\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t]/t^m;\mathbb{Q}) \ncong \pi_m\mathcal{L}^{\mathbb{Q}}_{\bigvee_{k=1}^{n}\bigvee_{\binom{n}{k}} S^k}(\mathbb{Q}[t]/t^m;\mathbb{Q}).$$

Proof. We consider the Tor-spectral sequence

$$\mathrm{Tor}_{*,*}^{\pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})}(\pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q}),\mathbb{Q}) \Rightarrow \pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t]/t^m;\mathbb{Q})$$

where the $\pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})$-module structure on $\pi_*\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})$ is induced by $t \mapsto t^m$. The m-chain complex $C^{(m)}_* := \mathcal{L}^{\mathbb{Q}}_{T^m}(\mathbb{Q}[t];\mathbb{Q})$ can be considered as an n-chain complex whose $m+1,\ldots,n$-coordinates are trivial. Then

$$C^{(m)}_* = \mathcal{L}^{\mathbb{Q}}_{T^m}(\mathbb{Q}[t];\mathbb{Q}) \hookrightarrow C^{(n)}_* := \mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})$$

is a sub-n-complex of $C^{(n)}_*$. We know that $\Delta_m \mapsto m!\mathsf{vol}_m$ in the homology of the total complex of $C^{(m)}_*$ and hence the same is true in $C^{(n)}_*$. Therefore the map

$$\pi_m\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q}) \to \pi_m\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t];\mathbb{Q})$$

that is induced by $t \mapsto t^m$ is nontrivial and by Lemma 4.5 the dimension of $\pi_m\mathcal{L}^{\mathbb{Q}}_{T^n}(\mathbb{Q}[t]/t^m;\mathbb{Q})$ is strictly smaller than the dimension of

$$\pi_m\mathcal{L}^{\mathbb{Q}}_{\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} S^k}(\mathbb{Q}[t]/t^m;\mathbb{Q}).$$

□

4.3. **Quotients by polynomials without constant term.** Let $q(t) = a_m t^m + \ldots + a_1 t \in \mathbb{Q}[t]$. Then we can still write $\mathbb{Q}[t]/q(t)$ as a pushout

$$\begin{tikzcd} \mathbb{Q}[t] \arrow[r, "t\mapsto q(t)"] \arrow[d, "t\mapsto 0"'] & \mathbb{Q}[t] \arrow[d, dotted] \\ \mathbb{Q} \arrow[r, dotted] & \mathbb{Q}[t]/q(t) \end{tikzcd}$$

hence the above methods carry over.

Proposition 4.17. *Let m_0 be the smallest natural number with $1 \leqslant m_0 \leqslant m$ with $a_{m_0} \neq 0$. Then*

$$\pi_{m_0}\mathcal{L}^{\mathbb{Q}}_{T^{m_0}}(\mathbb{Q}[t]/q(t);\mathbb{Q}) \ncong \pi_{m_0}\mathcal{L}^{\mathbb{Q}}_{\bigvee_{k=1}^{m_0} \bigvee_{\binom{m_0}{k}} S^k}(\mathbb{Q}[t]/q(t);\mathbb{Q}).$$

Proof. If $m_0 = 1$, then $\varepsilon t \in \mathsf{HH}^{\mathbb{Q}}_1(\mathbb{Q}[t];\mathbb{Q})$ maps to $\varepsilon(q(t)) \in \mathsf{HH}^{\mathbb{Q}}_1(\mathbb{Q}[t];\mathbb{Q})$ under the map $t \mapsto q(t)$. In the module of Kähler differentials this element corresponds to

$$a_1 dt + 2a_2 t dt + \ldots + m a_m t^{m-1} dt$$

but all these summands are null-homologous except for the first one. So $\varepsilon t \mapsto a_1 \varepsilon t \neq 0$ and this, along with Lemma 4.5, proves the claim.

We denote by $\Delta_{m_0}(q(t))$ the element $(q(t))_{\mathbf{1}_{m_0}}$. If $m_0 > 1$, then the diagonal element $\Delta_{m_0}(t)$ maps to

$$\Delta_{m_0}(q(t)) = \sum_{i=m_0}^{m} a_i \Delta_{m_0}(t^i)$$

and this is homologous to

$$(m_0)! a_{m_0} \mathsf{vol}_{m_0} + \text{ terms of higher } t\text{-degree}$$

by (4.12). Hence $\Delta_{m_0}(t)$ maps to a nontrivial element and again Lemma 4.5 gives the claim. □

4.4. **Truncated polynomial algebras in prime characteristic.** We know that for commutative Hopf algebras A over k the Loday construction is stable, so Loday constructions of truncated polynomial algebras of the form $\mathbb{F}_p[t]/t^{p^\ell}$ have the same homotopy groups when evaluated on an n-torus and on the corresponding bouquet of spheres. However, we show that there is a discrepancy for truncated polynomial algebras $\mathbb{F}_p[t]/t^n$ for $2 \leqslant n < p$.

Theorem 4.18. *Assume that $2 \leqslant n < p$ and $n \leqslant m$, then*

$$\pi_*(\mathcal{L}^{\mathbb{F}_p}_{T^m}(\mathbb{F}_p[t]/t^n; \mathbb{F}_p)) \not\cong \pi_*(\mathcal{L}^{\mathbb{F}_p}_{\bigvee_{k=1}^m \bigvee_{\binom{m}{k}} S^k}(\mathbb{F}_p[t]/t^n; \mathbb{F}_p)).$$

In particular for all $2 \leqslant n < p$ the pair $(\mathbb{F}_p[t]/t^n; \mathbb{F}_p)$ is not stable.

Proof. We consider the case $m = n$. The cases $n < m$ follow by an argument similar to that for Proposition 4.16.

As $\mathbb{F}_p[t]$ is smooth over $\mathbb{F}_p$, we know that $\mathbb{F}_p \to \mathbb{F}_p[t]$ is stable, so that

$$\pi_*(\mathcal{L}^{\mathbb{F}_p}_{T^n}(\mathbb{F}_p[t]; \mathbb{F}_p)) \cong \pi_*(\mathcal{L}^{\mathbb{F}_p}_{\bigvee_{k=1}^n \bigvee_{\binom{n}{k}} S^k}(\mathbb{F}_p[t]; \mathbb{F}_p)) \cong \bigotimes_{k=1}^n \bigotimes_{\binom{n}{k}} \mathsf{HH}^{[k],\mathbb{F}_p}_*(\mathbb{F}_p[t]; \mathbb{F}_p)$$

and $\mathsf{HH}^{[k],\mathbb{F}_p}_*(\mathbb{F}_p[t]; \mathbb{F}_p)$ is calculated in [4, §8] so that we obtain

$$\mathsf{HH}^{[k],\mathbb{F}_p}_*(\mathbb{F}_p[t]; \mathbb{F}_p) \cong B'_{k+1}$$

where $B'_1 = \mathbb{F}_p[t]$ and $B'_{k+1} = \mathrm{Tor}^{B'_k}_{*,*}(\mathbb{F}_p, \mathbb{F}_p)$ where the grading on $\mathrm{Tor}^{B'_k}_{*,*}(\mathbb{F}_p, \mathbb{F}_p)$ is the total grading. Thus in low degrees this gives $\mathsf{HH}^{\mathbb{F}_p}_*(\mathbb{F}_p[t]; \mathbb{F}_p) \cong \Lambda_{\mathbb{F}_p}(\varepsilon t)$ with $|\varepsilon t| = 1$, $\mathsf{HH}^{[2],\mathbb{F}_p}_*(\mathbb{F}_p[t]; \mathbb{F}_p) \cong \Gamma_{\mathbb{F}_p}(\varrho^0\varepsilon t)$ with $|\varrho^0\varepsilon t| = 2$. As $\Gamma_{\mathbb{F}_p}(\varrho^0\varepsilon t) \cong \bigotimes_{i\geqslant 0} \mathbb{F}_p[\varrho^k\varepsilon t]/(\varrho^k\varepsilon t)^p$ we can iterate the result.

Note that in $\mathsf{HH}^{[n],\mathbb{F}_p}_n(\mathbb{F}_p[t]; \mathbb{F}_p)$ there is always an indecomposable generator of the form $\varepsilon\varrho^0 \ldots \varrho^0\varepsilon t$ or $\varrho^0\varepsilon\varrho^0 \ldots \varrho^0\varepsilon t$ in degree n and we call this generator Δ_n. We also obtain a volume class

$$\mathsf{vol}_n := \varepsilon t_1 \cdot \ldots \cdot \varepsilon t_n \in \pi_n\mathcal{L}^{\mathbb{F}_p}_{S^1\vee\ldots\vee S^1}(\mathbb{F}_p[t]; \mathbb{F}_p) \hookrightarrow \pi_n\mathcal{L}^{\mathbb{F}_p}_{T^n}(\mathbb{F}_p[t]; \mathbb{F}_p).$$

The results from Proposition 4.11 work over the integers. If $n < p$, then $n!$ is invertible in $\mathbb{F}_p$ and therefore the class Δ_n maps to $n!\mathsf{vol}_n$. An argument analogous to Lemma 4.5 finishes the proof. □

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